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LETTER TO THE EDITOR

Vertex models on A–B square lattices

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Abstract. A specific vertex model is defined on a square lattice whose point set is partitioned into two subsets {A} and {B}. It is shown that if the A–B pattern of the partition does not contain elementary squares with an odd number of A points, then the vertex model is equivalent to an Ising model on a graph G. In order to construct the graph G, we make use of the properties of the A–B pattern on the square lattice. Several examples are examined and three different possibilities of behaviour are found, i.e., 'two-dimensional' Ising-type transitions, 'one-dimensional', and 'frozen in' behaviour.

An *A–B square lattice* is a square lattice of $N = N_A + N_B$ points whose point set is partitioned into a subset {A} of N_A points, and a subset {B} of N_B points. For each of the 2^N different A–B square lattices we may define the following vertex model:

$$Z_Q^V(e) = \sum_C \left(\prod_{i=1}^{N_A} w_{\xi(i)}\right)\left(\prod_{j=1}^{N_B} w'_{\xi(j)}\right), \qquad \begin{array}{l} w_1 = w_2 = w'_3 = w'_4 = w = \exp(-\beta e), \\ w_3 = w_4 = w'_1 = w'_2 = 0, \\ w_5 = w_6 = w'_5 = w'_6 = 1. \end{array} \tag{1}$$

The summation is extended over all possible six-vertex configurations C on the square lattice. The Boltzmann weights w_ξ and w'_ξ correspond to the six different kinds of vertices ($\xi = 1, 2, \ldots, 6$; figure 1) at the A and B points respectively, and Q denotes a specified A–B square lattice. If Q is the alternating A–B square lattice, then the vertex model (1) is equivalent to the (zero-field) Ising model on a square lattice with $N/2$ points and diagonal boundary conditions (Malakis 1979). The *alternating A–B square lattice* is an A–B square lattice in which every elementary square is an alternating A–B cycle (i.e. -A–B–A–B–). Let us relax this condition and define an *even A–B square lattice* by the restriction that every elementary square has an even number (0, 2, or 4) of A points. Henceforth, let Q denote any even A–B square lattice. We proceed to show that model (1) on Q is equivalent to an Ising model on a graph G which may be obtained from the A–B pattern on Q.

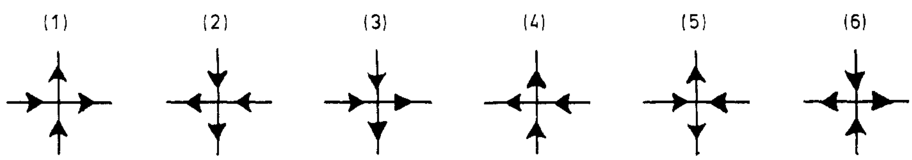


Figure 1. The six different vertices on the square lattice.

Let m and n be the numbers of rows and columns of Q. The number of unrestricted A–B arrangements on any row of the lattice is 2^n. However, once the A–B arrangement on some row is fixed (say AAABBA . . . AB), then there are only two A–B arrangements (AAABBA . . . AB and its conjugate BBBAAB . . . BA) on its adjacent rows, which will not produce squares with an odd number of A points. Thus, the number of even A–B square lattices is 2^{m+n-1}. For a given even A–B square lattice Q we can define a directed square lattice D.

Definition: D has the same sets of points and lines as Q; however, the lines of D are oriented. The vertical lines in the ith column of D are oriented down(up) if the top point of the ith column of Q is an A(B) point. The horizontal lines in the jth row($j = 1, 2, \ldots, m$) of D are oriented to the right if the leftmost point in the jth row of Q and the leftmost point in the top row($j = 1$) of Q are of the same type (both A or both B), otherwise they are oriented to the left. The square lattice digraph D has the following three properties.

Property (1). The lines incident to an A point of D (corresponding to an A point of Q) are oriented as in vertex (3) or (4), whereas the lines incident to a B point of D are oriented as in vertex (1) or (2) (compare the orientation at any B point in figures 2, 3(a) and 4(a) with the orientation of vertices (1) and (2) in figure 1).

Property (2). D has a unique anticycle partition, i.e., the oriented lines of D can be partitioned into a set of line-disjoint anticycles; an anticycle is a directed cycle in which adjacent lines have opposite directions (for an example see figure 2).

Property (3). The set of non-adjacent lines of an anticycle forms a dimer state of the anticycle; there are two such states. The lines of a dimer state are either all vertical (vertical dimer state) or all horizontal (horizontal dimer state).

For a given square lattice digraph D we can, using property (2), define a graph G.

Definition: With every anticycle of D we associate a point of G and two points of G are connected by a line of multiplicity t if the corresponding anticycles of D have t points in common.

From the definition of G it follows that $\Sigma_{l=1}^{L} t_l = N$, where L is the number of lines of G and t_l is the multiplicity of a line l (the number of points of G depends on the A–B pattern of Q). Let the points of G incident with a line l be denoted l_1 and l_2 and define spin variables on these points s_{l_1} and s_{l_2} $(=\pm 1)$. A spin configuration $\{s\}$ on G is obtained by specifying the values of all these variables ($l = 1, 2, \ldots, L$). It is suitable to define an Ising model on G by

$$Z_G^{\mathrm{I}}(e) = \sum_{\{s\}} \exp\left(-\tfrac{1}{2}\beta e \sum_{l=1}^{L} t_l(1 - s_{l_1}s_{l_2})\right). \tag{2}$$

The vertex model (1) on Q is equivalent to the Ising model (2) on G:

$$Z_Q^{\mathrm{V}}(e) = Z_G^{\mathrm{I}}(e). \tag{3}$$

In order to establish (3), we make use of the 'polymer' model on D (Malakis 1979). We may convert any six-vertex configuration on the square lattice into a bond graph by drawing a bond for each arrow pointing into an A point of the *alternating* A–B square lattice (see Lieb and Wu 1972). The bond graph forms a configuration on non-intersecting polygons that cover all lattice points. Since, for model (1), $w_3 = w_4 = w'_1 = w'_2 = 0$, some of the six-vertex configurations on Q have a zero weight. In consequence of property (1) of D, the bond graphs corresponding to non-zero terms of $Z^{\mathrm{V}}_{Q}(e)$ are compatible with the orientation of D, i.e., the non-intersecting polygons are also circuits of D. The correspondence between the configurations of circuits on D and the vertex configurations C' (with a non-zero weight) of model (1) on Q is one-to-one. Furthermore, the weight of a vertex configuration C' is w^k, where k is the number of vertices of type (1), (2), (3), or (4) of C'; k is also the number of polygonal corners of the corresponding configuration of circuits on D. Finally, we can, using property (3) of D, establish a one-to-one correspondence between the configurations of circuits on D and the spin configurations on G. This can be accomplished if we represent the $s = +1(-1)$ spin state of a point of G by the horizontal (vertical) dimer state of the anticycle of D corresponding to the point of G. The weight of a spin configuration $\{s\}$ on G is w^k, where $k = \frac{1}{2}\sum_{l=1}^{L} t_l(1 - s_{l_1}s_{l_2})$ is again the number of polygonal corners of the configuration of circuits on D corresponding to the spin configuration $\{s\}$ on G. Therefore, there exists a one-to-one correspondence between the non-zero terms of $Z^{\mathrm{V}}_{Q}(e)$ and the terms of $Z^{\mathrm{I}}_{G}(e)$, such that corresponding terms have equal weights. This completes the proof of (3).

As an application of (3), let us examine A–B patterns in which adjacent A(B) points form 'rectangles' with k_1 points vertically and k_2 points horizontally. Consider the following three cases.

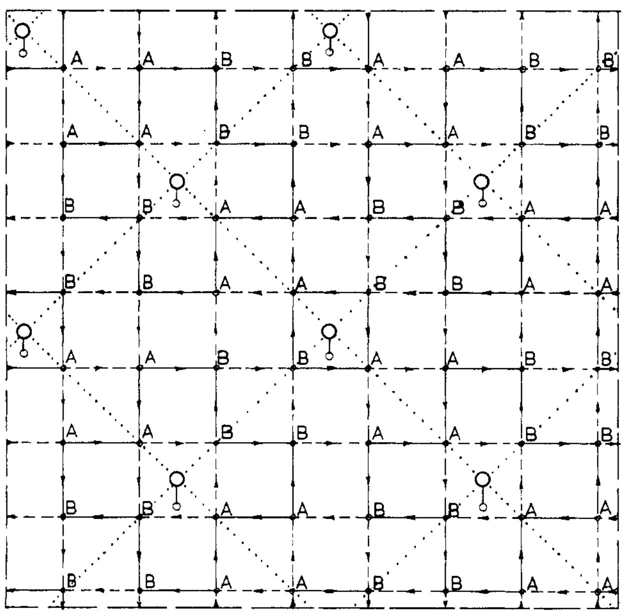
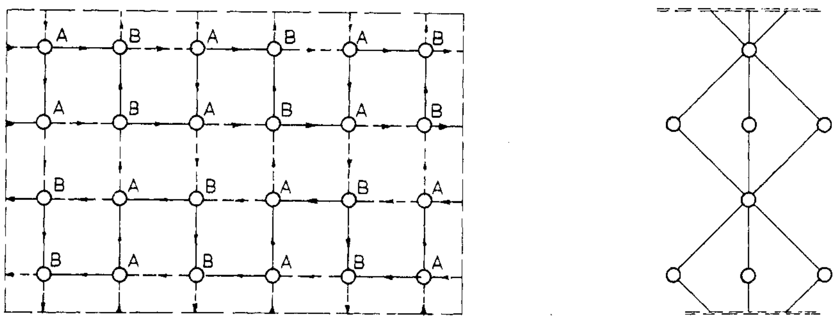


Figure 2. An A–B square lattice Q in which adjacent A(B) points form 2×2 'rectangles'. Arrows specify the orientation of D. The anticycle partition of D is illustrated by full and broken oriented lines. The lines of the graph G are the dotted lines (these lines form a square lattice with diagonal boundary conditions) and the small full lines (these lines connect each point of the 'diagonal' square lattice to a linear chain of $k_1 - 1 = 1$ points). The multiplicity of the dotted lines is 2, whereas the multiplicity of the small full lines is 4.

Case 1, $k_1 = k_2$. One can easily verify that model (1) on Q undergoes an Ising-type transition. The transition temperature T_c depends on k_1 and is located by $|\sinh(ek_1/kT_c)| = 1$. An example ($k_1 = k_2 = 2$) is given in figure 2. We note that the partition function $Z^{\mathrm{I}}_G(e)$ can be easily related to the partition function of the usual Ising model on a square lattice with diagonal boundary conditions.

Case 2, $k_1 \neq k_2$. The vertex model (1) on Q does not undergo a phase transition. This is suggested from the fact that G is 'one-dimensional' (for an appropriate definition of dimensionality, see Kasteleyn 1963). An example is given in figure 3 ($k_1 = 2$, $k_2 = 1$). The partition function can be calculated using the transfer matrix method; in the thermodynamic limit we find $\lim_{m,n\to\infty}(1/mn)\ln(Z^{\mathrm{V}}_Q) = \frac{1}{4}\,ln(1+w^4)$.

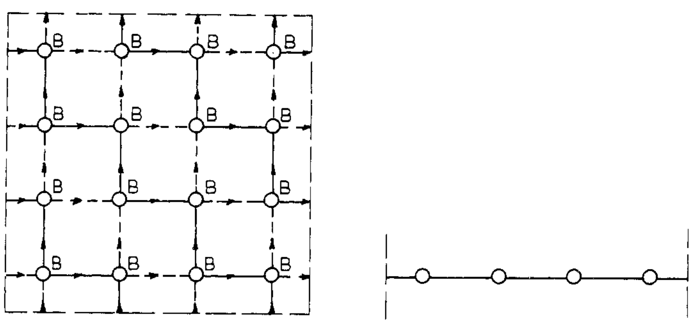


(*a*) (*b*)

Figure 3. (*a*) An A–B square lattice Q in which adjacent A(B) points form 2×1 'rectangles'. (*b*) The graph G. The multiplicity of its lines is 2.

Case 3, $k_1 = k_2 = m = n$. The vertex model (1) on Q is equivalent to the Ising model on a simple linear chain of m points (figure 4). However, in the appropriate thermodynamic limit, model (1) will show 'frozen in' behaviour, because at all temperatures the entropy per point (atom) is zero (the number of configurations is 2^m, whereas the number of points is m^2). We also note that the 'polymer' model on D, in this case, is the same as Nagle's 'dimer model B to a polymer model' at maximum density (Nagle 1974).

We guess that these three cases contain all possibilities of behaviour of model (1) on Q. Furthermore, since different parts of the A–B pattern on Q may satisfy case 1 with



(*a*) (*b*)

Figure 4. (*a*) A square lattice in which all points are B points. (*b*) The graph G is a simple linear chain. The multiplicity of its lines is m.

different values of k_1, multiple transitions are possible. The generalisation of (3) to an arbitrary A–B square lattice is not evident because property (1) of D is a consequence of the fact that Q is an even A–B square lattice. Finally, we point out that for $e > 0$ model (1) is antiferroelectric, and its Ising equivalent is ferromagnetic.

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